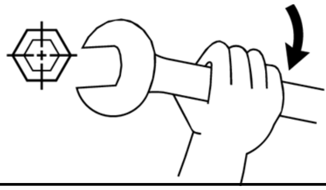
Michael Gaida (Pixabay)

Equilibrium

- An object is in equilibrium if it is stationary or moving with a constant velocity.
- There are two conditions necessary for a system to achieve equilibrium.
  - Net external force on the system must be zero (translational equilibrium)
  - Net external torque on the system must be zero (rotational equilibrium).

Torque

- Torque is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time).
  - It is the rotational equivalent of a force.



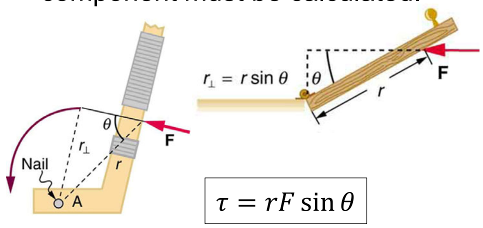
Jeff Stein (Pixabay)

- The rotation is around a pivot point.
- The distance from the pivot point to the point where the force acts is called the lever arm or moment arm.
- The applied force must be perpendicular to the lever arm to cause rotation.

- The magnitude of torque is defined to be

$$\boxed{\tau = r_{\perp} F}$$

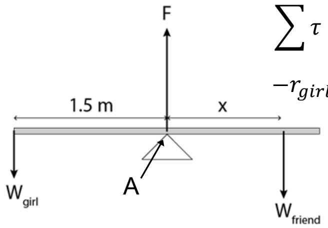
- If the force is not perpendicular to the lever arm, then the perpendicular component must be calculated.



$$\boxed{\tau = rF\sin\theta}$$

Example

A 50.0 kg girl is sitting at the end of a 3.0 m long see-saw. How far from the center must her 60.0 kg sit such that the see-saw is in equilibrium?



$$\sum \tau = 0$$ about A

$$-r_{girl}F_{girl} + r_{friend}F_{friend} = 0$$

$$r_{friend} = \frac{r_{girl}m_{girl}g}{m_{friend}g}$$

$$r_{friend} = \frac{(1.5)(50)}{(60)} = 1.3\text{ mN}$$

Angular Acceleration

- Angular acceleration is defined as the rate of change of angular speed.

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Units: rad/s^2

- Angular acceleration is related to translational acceleration.

$$a = \alpha r$$

Kinematics of Rotational Motion

- Kinematics for rotational motion is completely analogous to translational kinematics.

Rotational	Translational
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$
$\omega = \omega_0 + \alpha t$	$v_x = v_{x0} + a_x t$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$

Example

A wheel is rotated from rest with an angular acceleration of 8.0 rad/s^2. It accelerates for 5.0 s. Calculate

a) the angular speed.

b) the number of revolutions that the wheel has rotated through.

a) $\omega = \omega_0 + \alpha t$

$\omega = (8)(5) = 40 \text{ rad/s}$

b) $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

$\theta = \frac{1}{2}(8)(5)^2 = 100 \text{ rad}$

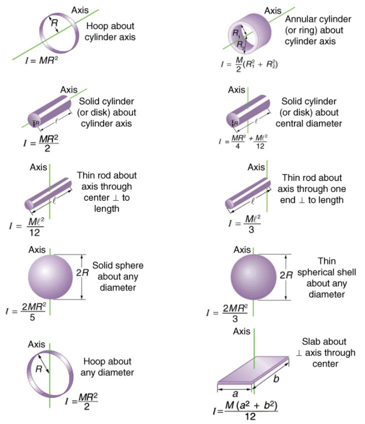
$\text{rotations} = \frac{100}{2\pi} = 16$

Moment of Inertia

- The moment of inertia, I, of an object is defined as the sum of mr^2 for all the point masses of which it is composed.

$$I = \sum mr^2$$

- Moment of inertia is analogous to mass in translational motion.
- The moment of inertia for any object depends on the chosen axis.
- Units: kg·m^2

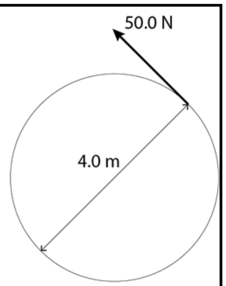


- The general relationship among torque, moment of inertia, and angular acceleration is

$$\boxed{\vec{a} = \frac{\sum \vec{\tau}}{I} = \frac{\vec{\tau}_{net}}{I}}$$

Example

A 45 kg child is sitting on the edge of a 60.0 kg merry-go-round with a diameter of 4.0 m for one rotation. A constant force of 50.0 N is applied tangentially to the edge. The moment of inertia is $\frac{1}{2}mr^2$. Calculate the angular speed of the merry-go-round.



$$\vec{\alpha} = \frac{\vec{\tau}_{net}}{I} \qquad \omega^2 = \omega_0^2 + 2\alpha\theta \qquad \tau = rF\sin\theta$$

$$\alpha = \frac{\omega^2}{2\theta}$$

$$I_m = \frac{1}{2}mr^2 \qquad I_c = mr^2$$

$$I_{total} = \frac{1}{2}m_m r^2 + m_c r^2$$

$$\frac{\omega^2}{2\theta} = \frac{rF}{\frac{1}{2}m_m r^2 + m_c r^2}$$

$$\omega = \sqrt{\frac{2\theta rF}{\frac{1}{2}m_m r^2 + m_c r^2}}$$

$$\omega = \sqrt{\frac{2\theta rF}{\frac{1}{2}m_m r^2 + m_c r^2}}$$

$$\omega = \sqrt{\frac{2(2\pi)(2)(50)}{\frac{1}{2}(60)(2)^2 + (45)(2)^2}} = 1.8 \text{ rad/s}$$

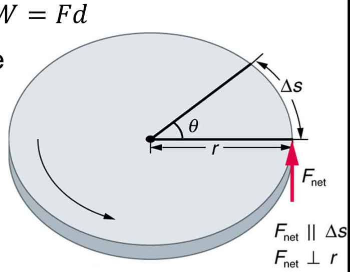
Rotational Kinetic Energy

- Work must be done to rotate objects.

$$W = Fd$$

- When a net force is applied perpendicular to the lever arm, the object rotates through an arc length Δs.

$$W = F\Delta s$$



$\tau = rF$ $\quad W = \tau \frac{\Delta s}{r}$ $\quad \theta = \frac{\Delta s}{r}$

$W = \tau\theta$ $\quad \tau = I\alpha$

$W = I\alpha\theta$ $\quad \omega^2 = \omega_0^2 + 2\alpha\theta$

$$\alpha\theta = \frac{(\omega^2 - \omega_0^2)}{2}$$

$$W = \frac{1}{2} I\omega^2 - \frac{1}{2} I\omega_0^2$$

- This is the work-energy theorem for rotational motion.

- Through an analogy with translational motion, we define the term $\frac{1}{2} I\omega^2$ to be rotational kinetic energy for an object with a moment of inertia I and an angular velocity ω.

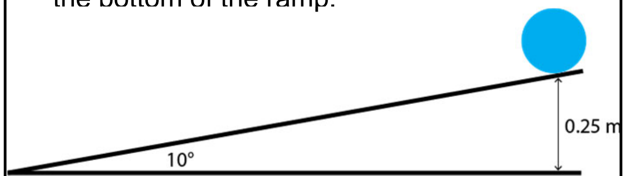
$$\boxed{K = \frac{1}{2} I\omega^2}$$

Rolling and Slipping

- If an object makes a perfectly frictionless contact with a surface it is impossible for the object to roll – it simply slides.
- When there is friction the object can roll.
- Since the point of contact between the rolling body and the surface on which it rolls is instantaneously stationary, the coefficient of static friction should be used in calculations involving rolling.
- The point of contact must be stationary because it does not slide.

Example

A ball rolls down a ramp as shown without slipping. The moment of inertia of the ball is $\frac{2}{3}mr^2$. Calculate the linear velocity at the bottom of the ramp.



Energy is conserved.

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 \qquad v = \omega r$$

$$mgh = \frac{1}{2}\left(\frac{2}{3}mr^2\right)\left(\frac{v}{r}\right)^2 + \frac{1}{2}mv^2$$

$$gh = \frac{v^2}{3} + \frac{v^2}{2} = \frac{5v^2}{6}$$

$$v = \sqrt{\frac{6gh}{5}}$$

$$v = \sqrt{\frac{6(9.8)(0.25)}{5}} = 1.7\ \mathrm{m/s}$$

Angular Momentum

- Angular momentum is defined as the product of the moment of inertia and the angular velocity.

$$\boxed{L = I\omega}$$

Units: kg m^2 rad s^{-1}

- A torque is required to make an object rotate.
- If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases.
- The greater the net torque, the more rapid the increase in L.
- The relationship between torque and angular momentum is

$$\boxed{\Delta L = \tau \Delta t}$$

(This is the rotational form of Newton's second law.)

Conservation of Angular Momentum

- The total angular momentum of a system remains constant providing no external torque acts on it.

Example

Suppose a star of radius R_1 has a period of 20 days. The star collapses to a radius R_2 which is smaller by a factor of 10000 without losing mass. What is the new period of the star?

Moment of inertia of a sphere = $\frac{2}{5}mr^2$

Angular momentum is conserved.

$$L_1 = L_2$$

$$I_1\omega_1 = I_2\omega_2$$

$$I = \frac{2}{5}mr^2 \qquad \omega = \frac{2\pi}{T}$$

$$\frac{2}{5}mR_1^2\left(\frac{2\pi}{T_1}\right) = \frac{2}{5}mR_2^2\left(\frac{2\pi}{T_2}\right)$$

$$R_1 = 10000R_2$$

$$T_2 = T_1\left(\frac{R_2}{10000R_2}\right)^2$$

$$T_2 = 20\left(\frac{1}{1\times 10^4}\right)^2 = 2\times 10^{-7}\text{ days} = 0.017\text{ s}$$